contributions

The Transient Response of Linear Antennas and Loops*

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Summary—The transient response of straight wires and circular loops when short pulses are applied is studied experimentally and theoretically. It is shown that the initial response is always that of an infinitely long antenna at a frequency near the upper limit of the frequencies contained in the pulse provided this is sufficiently short so that the first reflection from the end of the wire or loop is not superimposed on it.

INTRODUCTION

If a LINEAR dipole or a loop antenna is connected by means of a transmission line to an oscillator operating at a fixed angular frequency ω, the circuit characteristics of the antenna are entirely described by its apparent input impedance.[1] For nonsteady-state operation, if transient signals such as pulses or steps in voltage are applied to the driving transmission line the concept of the input impedance loses its significance, except that in principle the transient response may be computed by superposition of the responses to the individual frequency components of the Fourier representation of the applied signal.[2]

The circuit characteristics of dipole antennas and circular loop antennas for transient pulses which are short in time compared to the propagation time of the disturbance along the half-length h or the half-circumference l have been investigated experimentally. The antennas consisted of $\frac{3}{16}$ inch OD brass tubing with the linear dimension h or l of 9 feet. They were mounted on a 48×24 foot metallic ground screen and driven by a coaxial cable with a characteristic resistance $R_c = 50$ ohms. Short dc pulses with variable length between 1 and 50 mμsec were generated by the discharge through a mercury switch of short open-ended sections of coaxial cable. A small part of the energy is taken out to trigger the sweep of a Tektronix 517 high-speed oscilloscope (Fig. 1). The main pulse is led through a 100-foot length of RG 58/U cable to the antenna. For the observation of the reflected pulses, a fraction of the signal is coupled out in a coaxial Tee placed midway between the pulse generator and the antenna. The signal is led through a 50-foot section of RG 63/U cable with a characteristic resistance of 120 ohms directly to the vertical deflection plates of the CRT.

The length of the connecting transmission lines is sufficient to assure that the first images due to the mismatch at the T section and at the coupling to the scope plates are delayed by about 150 mμsec. The pulses reflected from the open-ended transmission line, with no antenna connected, are shown in Fig. 2. The pulse durations are 3 and 7 mμsec, respectively; the rise time is initially of the order of 1 mμsec, but when shown on the scope it has increased to about 2 mμsec due to the long

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[1] R. W. P. King, "The Theory of Linear Antennas," Harvard University Press, Cambridge, Mass.; 1956.

[2] H. J. Schmitt, "Transients in cylindrical antennas," *Proc. IEE*, pt. C, Monograph No. 377 E; April, 1960.

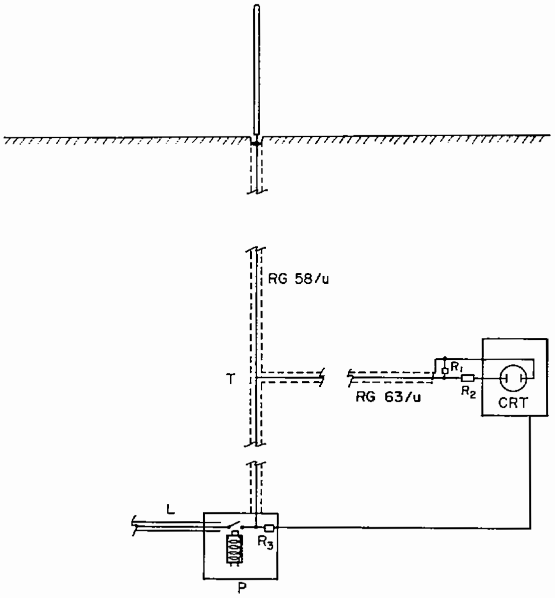

Fig. 1—Experimental arrangement for measuring transient responses from linear antennas and loops.

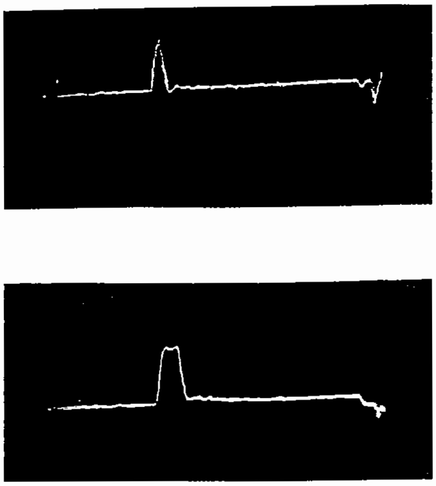

Fig. 2—Reflected pulses from open-ended transmission line. Pulse duration 3 mμsec and 7 mμsec. Sweep rate 10 mμsec/cm.

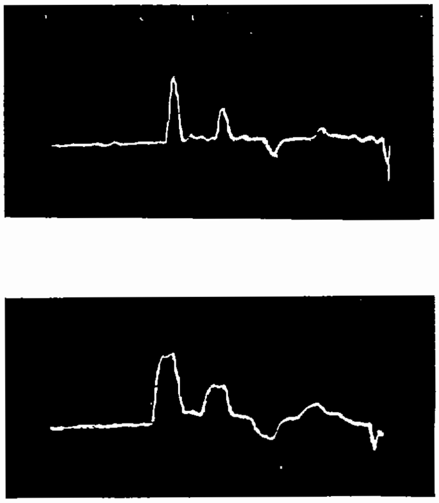

Fig. 3—Reflection of short pulses from monopole antenna with half length $h=9'$. Initial pulse duration 3 mμsec and 7 mμsec. Sweep rate 10 mμsec/cm.

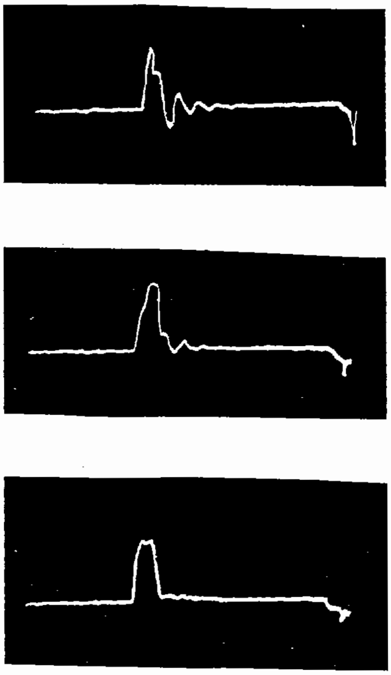

Fig. 4—Top and center, reflection of short pulses from monopole antenna with length $h=50$ cm; bottom, reflection from open-ended line of same voltage pulse as in middle figure. Pulse duration 3 mμsec for top, 7 mμsec for center and bottom figures.

sections of cable. The slight ringing is due to the inductance of the lead-off wires from the plates of the oscilloscope in conjunction with the plate capacitance.

TRANSIENTS IN LINEAR ANTENNAS

The response of a 9-foot linear monopole to both the shorter and longer pulses is shown in Fig. 3. The behavior of the return signal as a function of time closely resembles that of a section of open-ended transmission line of equal length. After the initial reflection from the junction point between the antenna and the driving line further pulses are observed each time the disturbance has traveled with the velocity of light c to the tip of the antenna and back to the driving point, in this case after time intervals of about 18 mμsec. The initial reflection from the antenna is like that from a predominantly resistive termination with a resistance $R>R_c$, where R is the apparent resistance of the antenna and $R_c=50$ ohms is the characteristic resistance of the line. A comparison of the measured amplitude of the first reflected pulse when the line is terminated in the antenna with the amplitude of the pulse reflected from the same line with its end open, yields a reflection coefficient $r=0.72$ for the antenna. This corresponds in some manner to an impedance Z such that with $R_c=50$ ohms

$$Z = R_c \frac{1+r}{1-r} = 307.1 \text{ ohms}.$$

Evidently, the tip of the antenna acts in the sense of an open end since the pulse reflected at this point is in phase with the incident pulse. Upon reaching the driving point the pulse splits into two parts, the transmitted pulse (second pulse in Fig. 3) and a pulse reflected at this discontinuity. The phase of the pulse reflected at the driving point is inverted as for a reflection at a load with resistance smaller than R. After traveling back to the tip of the antenna a second time and subsequently back to the driving point, the pulse is observed as the third one in Fig. 3. The phases of succeeding pulses alternate. The initial response of a much shorter antenna ($h=0.5$ m) is the same as that of a longer antenna as may be seen in Fig. 4. Owing to the shorter length, the reflected pulses arrive in closer succession than on a longer antenna. The general pattern of the reflected pulses is greatly affected by the ratio of the apparent resistance R of the antenna to the characteristic impedance R_c of the driving transmission line. In particular, if R_c were chosen approximately equal to the apparent impedance of the antenna so that $r \doteq 0$, only the direct reflection from the end of the dipole would have a significant amplitude.

It is interesting to note that if the pulse duration exceeds $2h/c$ the positive reflection from the antenna tip is superimposed on the initial reflection and the incident pulse at the driving point. This leads to peak amplitudes in the driving line that exceed the amplitude of the incident pulse by a factor of up to 2.25. This has to be taken into account for high-power systems in which a breakdown may occur.

IMPEDANCE OF AN INFINITELY LONG ANTENNA

The question now arises, what is the significance of the impedance $|Z|=307.1$ ohms that apparently terminates the line and that corresponds to a reflection coefficient 0.72? In the frequency domain a sharp rectangular pulse is composed of a very broad spectrum of frequencies for *each* of which a cylindrical antenna of given length and radius has a *different* impedance. Which of this wide range of impedances corresponds to the particular value that determines the reflection coefficient observed when a pulse is applied to the antenna?

Since the impedance of an antenna of finite length depends on that length, whereas the response of an antenna to a short pulse must be independent of the length during the initial instants before the first reflection from the end arrives at the driving point, it follows that only the impedance of infinitely long antennas can be involved since this is independent of reflections from the end.

The admittance of an infinitely long antenna at a single angular frequency ω may be obtained from a formula derived by Wu[3] for very long antennas. In the limit as the length of the antenna becomes infinite, Wu's formula reduces to the following form for a base-driven antenna over a ground screen:

$$Y(\omega) = \frac{2j}{\zeta_0}\left[\ln\left(1 - j\frac{\pi}{\Omega}\right) + \frac{\pi^2}{12}\left(\frac{1}{D^2} - \frac{1}{(D - j\pi)}\right)\right], \quad (1)$$

where

$$\Omega = \ln\left(\frac{1}{ka}\right) - \gamma = \ln\left(\frac{1}{\delta ka}\right) \quad (2)$$

$$D = \Omega - \ln 2 \quad (3)$$

and $\gamma = 0.5772\cdots$, $\delta = e^\gamma = 1.781$, $\zeta_0 = 120\pi$ ohms, $k = \omega/c = 2\pi/\lambda$. If the real and imaginary parts are separated, the conductance and susceptance for $Y = G + jB$ are

$$G(\omega) = \frac{2}{\zeta_0}\left[\tan^{-1}\frac{\pi}{\Omega} + \frac{\pi^3 D}{6(D^2 + \pi^2)^2}\right] \quad (4)$$

$$B(\omega) = \frac{1}{\zeta_0}\left[\ln\left(1 + \frac{\pi^2}{\Omega^2}\right) + \frac{\pi^4(3D^2 + \pi^2)}{6D^2(D^2 + \pi^2)^2}\right]. \quad (5)$$

An approximate expression for the admittance of an infinitely long center-driven antenna when ka is sufficiently small so that the inequality

$$\left(\frac{\pi}{\Omega}\right)^2 \ll 1 \quad (6)$$

is satisfied is given by the leading terms in (4) and (5). Thus,

$$G(\omega) \doteq \frac{2\pi}{\zeta_0 \Omega}; \qquad B(\omega) \doteq \frac{\pi^2}{\zeta_0 \Omega^2}. \quad (7)$$

This formula for the conductance is essentially the same as that derived by Papas.[4]

The impedance of the base-driven antenna is

$$Z(\omega) = R(\omega) + jX(\omega) = \frac{G(\omega)}{G^2(\omega) + B^2(\omega)} - j\frac{B(\omega)}{G^2(\omega) + B^2(\omega)}, \quad (8)$$

where $G(\omega)$ and $B(\omega)$ are given by (4) and (5). When (6) is satisfied the approximate impedance is

$$Z(\omega) \doteq \frac{\zeta_0}{2\pi}\Omega - j\frac{\zeta_0}{4}, \quad (9)$$

so that

$$R(\omega) \doteq \frac{\zeta_0}{2\pi}\left[\ln\left(\frac{1}{ka}\right) - 0.5772\right]$$

$$= 60\left[\ln\frac{c}{\omega a} - 0.5772\right] \text{ohms} \quad (10)$$

$$X(\omega) \doteq X = -\zeta_0/4 = -30\pi \text{ ohms}. \quad (11)$$

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[3] T. T. Wu, "Theory of the Dipole Antenna and the Two-Wire Transmission Line," Cruft Lab., Harvard Univ., Cambridge, Mass., Tech. Rept. No. 318, March, 1960; also in *J. Math. Phys.*, vol. 2, pp. 550–574, July–August, 1961.

[4] C. H. Papas, "On the Infinitely Long Cylindrical Antenna," Cruft Lab., Harvard University, Cambridge, Mass., Tech. Rept. No. 58, September, 1948; also in *J. Appl. Phys.*, vol. 20, pp. 437–440, May, 1949.

The impedance of a center-driven antenna is twice the value in (9). Note that when (6) is satisfied, the reactance of the infinitely long base-driven antenna is essentially independent of the frequency and equal to −94.2 ohms; moreover, when $R(\omega)$ is well approximated by (10), it varies only slowly with frequency. Curves for $R(\omega)$ and $X(\omega)$ for a base-driven antenna over a ground plane as determined from (8) with (4) and (5) and from the approximate formulas (10) and (11) are shown in Fig. 5 over a wide range of values of $(ka)^{-1}$. The corresponding reflection coefficient $r = |r| e^{j\phi}$ when the antenna terminates a 50 ohm line is also shown. It is clear that (10) and (11) are quite good approximations.

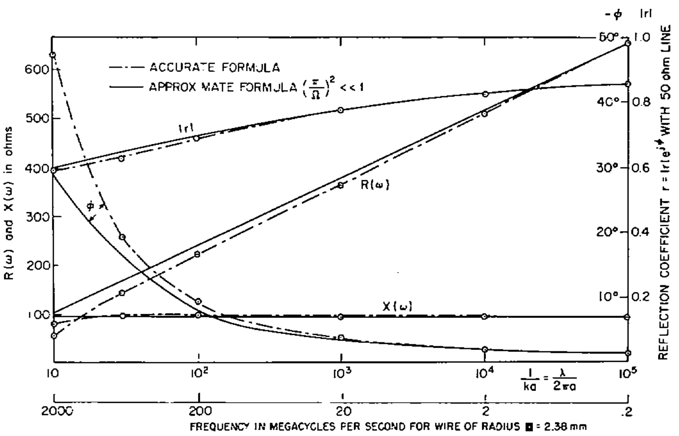

Fig. 5—Impedance $Z(\omega) = R(\omega) + jX(\omega)$ and reflection coefficient $r = |r| e^{j\phi}$ of base-driven infinitely long wire above a ground screen. (R_c of driving line, 50 ohms.)

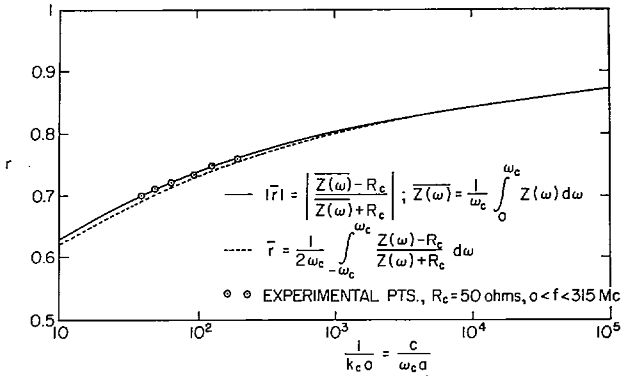

Fig. 6—Reflection coefficient of antenna terminating line for pulse.

Let a short voltage pulse that has a finite rise time so that its upper frequency limit is $\omega_c/2\pi$, be applied to an antenna of radius a. If ω_c and a have such values that for all frequencies in the pulse (6) is satisfied, that is,

$$\pi^2 \ll \left|\ln \frac{c}{\omega a}\right|^2, \tag{12}$$

the initial response of the antenna is given approximately by the average impedance $\overline{Z(\omega)}$ for the range of frequencies in the pulse. Specifically, if ω_c is the upper angular frequency limit in the pulse,

$$Z = \overline{Z(\omega)} = \frac{1}{\omega_c}\int_0^{\omega_c} S(\omega) Z(\omega) d\omega, \tag{13}$$

where $S(\omega)$ is the spectrum function for the pulse. As an approximation let it be assumed that $S(\omega) \doteq 1$ over the range $0 \leqq \omega \leqq \omega_c$. In this case

$$\begin{aligned} \overline{Z(\omega)} &= \frac{\zeta_0}{2\pi\omega_c}\int_0^{\omega_c}\left[\ln\frac{c}{\omega a} - 0.5772 - j\frac{\pi}{2}\right] d\omega \\ &= \frac{\zeta_0}{2\pi}\left(1 + \ln\frac{c}{\omega_c a} - 0.5772 - j\frac{\pi}{2}\right). \end{aligned} \tag{14}$$

Thus, the average impedance is approximated by an expression that involves only the impedance at the upper frequency limit. That is,

$$\overline{Z(\omega)} = Z(\omega_c) + \frac{\zeta_0}{2\pi}\cdot \tag{15}$$

The magnitude of the reflection coefficient

$$|\bar r| = \left|\frac{\overline{Z(\omega)} - R_c}{\overline{Z(\omega)} + R_c}\right| \tag{16}$$

is represented by the solid curve in Fig. 6 over a range of values of

$$(k_c a)^{-1} = \frac{\lambda_c}{2\pi a} \quad \text{where} \quad k_c = \frac{\omega_c}{c}\cdot$$

An alternative formulation[5] is to define the average reflection coefficient directly in the form

$$\begin{aligned} \bar r &= \frac{1}{2\omega_c}\int_{-\omega_c}^{\omega_c} \frac{Z(\omega) - R_c}{Z(\omega) + R_c} d\omega \\ &= \frac{1}{\omega_c}\int_0^{\omega_c} \frac{Z(\omega)Z(-\omega) - R_c^2}{[Z(\omega) + R_c][Z(-\omega) + R_c]} d\omega. \end{aligned} \tag{17}$$

With the notation (2) in the form

$$\Omega = \ln\left(\frac{1}{\delta k a}\right), \qquad \delta = e^\gamma = 1.781 \tag{18}$$

and the short hands

$$R_c = \frac{\zeta_0}{2\pi} d, \qquad d = \ln\frac{b}{a}; \qquad C = \frac{\pi}{a}, \tag{19}$$

it follows that

$$Z(\omega) = \frac{\zeta_0}{2\pi}(\Omega - jC), \qquad Z(-\omega) = \frac{\zeta_0}{2\pi}(\Omega + jC). \tag{20}$$

Hence,

$$\bar r = \frac{1}{\delta k_c a}\int_{\Omega_c}^{\infty} \frac{\Omega^2 + C^2 - d^2}{(\Omega + d)^2 + C^2} e^{-\Omega} d\Omega. \tag{21}$$

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[5] Suggested by the referee.

With the change in variable, $x=\Omega+d$, it follows that

$$\bar{r} = \frac{e^d}{\delta k_c a}\int_{\Omega_c+d}^{\infty}\left[1 - \frac{2dx}{x^2 + C^2}\right]e^{-x}dx. \tag{22}$$

The first term in (22) may be integrated directly so that

$$\bar{r} = 1 - \frac{de^d}{\delta k_c a}\int_{\Omega_c+d}^{\infty}\frac{2x}{x^2 + C^2}\,e^{-x}dx. \tag{23}$$

The remaining integral in (23) may be expanded by factoring the denominator and expressed in terms of the exponential integral with complex arguments. Although tables of these functions are available,[6] they are of limited range and not adequate for the evaluation of $\bar{r}$. Since the approximation (6) has already been imposed, no further restriction is implied if it is assumed in (23) that $C^2/x^2=\pi^2/4(\Omega+d)^2\ll 1$. Note that for $\Omega_c+d\gtrsim 3$, $C^4/x^4\lesssim 0.06$.

If the integrand in (23) is expanded in powers of C/x, and terms of orders higher than C^2/x^2 are neglected, the following result may be obtained directly:

$$\bar{r} \doteq 1 + \frac{2d\left(1 - \frac{C^2}{2}\right)}{\delta k_c a}\,e^d Ei(-\Omega_c - d) - C^2 d\left[\frac{2}{\Omega_c + d} - \frac{1}{(\Omega_c + d)^2}\right], \tag{24}$$

where

$$-Ei(-x) = \int_x^{\infty}\frac{e^{-x}}{x}\,dx$$

is the tabulated exponential integral with real argument; also $C=\pi/2$, $\delta=1.781$, $d=\ln b/a$ and $\Omega_c=\ln(1/\delta k_c a)$. The value of $\bar{r}$ computed from (24) is shown in broken line in Fig. 6. It is seen to differ only slightly from $\bar{r}$ computed from (16). It is obvious that this must be true in the case at hand since over the range of interest $R_c=50$ ohms is quite small compared with $|Z(\omega)|$. Although the definition (17) of the average reflection coefficient is theoretically preferable, the form (16) in terms of the average impedance is analytically simpler and practically convenient.

Experimentally determined values of r, including the value 0.72 to which reference has already been made, are also shown in Fig. 6. The short, approximately rectangular pulse applied to the antenna had an upper frequency limit of approximately 315 Mc; this corresponds to $\lambda_c=0.95$ m. The six experimental points are for antennas with six different values of a. The agreement with the theoretical curve is seen to be excellent.

Transient Response of Two Parallel Conductors: Transmission Line and Antenna

When a short pulse is applied to a two-wire line or a single-wire line over an image plane, the initial response must be that of an infinitely long line which has an input resistance equal to the characteristic resistance R_{c1} of the line. The curve in Fig. 7 shows the reflection coefficient

$$|r| = \left|\frac{R_{c1} - 50}{R_{c1} + 50}\right|$$

of a single-conductor line of radius $a=2.38$ mm at a distance $b/2$ from an image plane when connected as the terminating load for a 50 ohm coaxial line. Since $R_{c1}=(\zeta_0/2\pi)\ln b/a$ approaches infinity as b/a increases without limit, $|r|$ must approach 1. However, two parallel conductors behave like a transmission line only so long as they are sufficiently close together so that a positive pulse traveling along the one conductor is influenced by the presence of the similar negative pulse traveling in the second conductor. This means that the length of the pulse must be long compared with the time of travel with the velocity of light of a signal from the pulse on the one conductor to the pulse on the other. If the two conductors are separated further, the presence of the pulse in the second one can have no effect on the pulse in the first one and it must behave just as if isolated, that is, like an antenna. Hence, if a pulse is applied to a conductor placed sufficiently close to an image plane, the initial response is determined by the characteristic resistance of the transmission line, formed by the conductor and the image plane. If the image plane is moved further and further away, the response continues to be determined by the increasing characteristic resistance of the line until the separation is sufficient to make it impossible for the effect of the pulse of

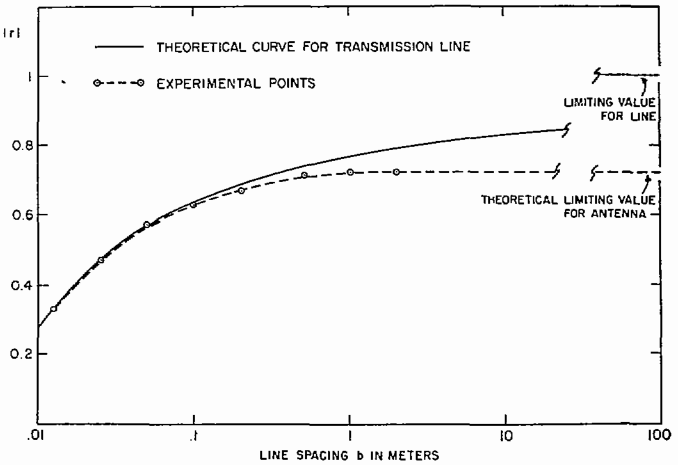

Fig. 7—Magnitude of reflection coefficient $|r|$ of two-wire line terminating a 50-ohm coaxial line. Radius of conductors, 2.38 mm; distance between centers, b.

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[6] For example, W. E. Bleick, "Tables of Associated Sine and Cosine Integral Functions and of Related Complex-Valued Functions," Navy Dept., Bureau of Ships, Monterey, Calif., Tech. Rept. No. 10, Project Order 10731/52, May, 1953.

the charges in the second conductor to reach the corresponding pulse on the first conductor. The response is then that of an antenna of radius a with an impedance given by (15). The reflection coefficient of the conductor as a termination for a coaxial line follows the curve in Fig. 7 only so long as the image plane is close enough. It must diverge from this curve and remain constant at the value of $|\bar{r}|$ given by (16) as soon as the image plane is sufficiently far away. The reflection coefficient measured on a coaxial line when terminated in a vertical conductor over a large horizontal ground screen when a vertical image plane is moved from a point very close to the conductor to infinity is also shown in Fig. 7. The radius of the conductor was 2.38 mm, the highest frequency in the pulse 315 Mc, so that the upper limit of the reflection coefficient for the conductor is its value as an antenna, *viz.*, 0.72. This point is shown on the right in Fig. 7. The measured curve behaves exactly as predicted.

TRANSIENTS IN LOOP ANTENNAS

As shown in Fig. 8 the response to a pulse by a circular half-loop above a ground screen corresponds very closely to the behavior of a short-circuited section of transmission line with a length corresponding to the half-circumference of the loop. Since at the first instant when the transient signal reaches the driving point of the loop antenna, no significant difference exists between a linear dipole and a loop, the initial response is also that of an impedance given by (8) or (9) for a $\frac{3}{16}$ inch OD wire, as measured from the height of the initial reflection. This value is essentially determined by the thickness of the antenna conductor alone and the upper frequency limit of the pulse. Successive reflections resemble those on a short-circuited transmission line. Due to the phase reversal of the pulse on each of the subsequent reflections at the center of the loop opposite the driving point and at the driving point itself, all further pulse images observed in the driving transmission line appear with negative phase. Although at least four following images are clearly separated in time, the distortion is somewhat larger than in the case of the linear dipole. Note, however, that the reflection from the opposite end of the driving point is negative in the case of the loop, so that if this reflection is superimposed on an incident pulse of sufficiently long duration a reduction rather than an increase in the return signal on the driving line occurs.

The first information at the driving point to show that the attached wires belong to a closed loop may be expected to arrive with a time delay of only $t_1 = 2d/c \doteq 11$ mμsec, where d is the diameter of the circular loop. Experimentally, however, it is found that the only significant contributions arrive after $t_1' = \pi d/c$. This indicates that the effect of reradiation from the opposite side of the loop is negligible in its direct contribution to the electromagnetic field in the vicinity of the driving point during the first instants of time.

RADIATION OF TRANSIENT SIGNALS FROM A DIPOLE

If a short pulse is applied at the terminals of a center-driven antenna, pulse-shaped distributions of positive and negative charges proceed in the two branches of the antenna towards the ends. They are equivalent to two pulse-shaped sheets of surface current. The far-zone field of the antenna is proportional to the time derivative of the local current density. At any instant of time, except when the short pulses are close to the ends of the antenna, this consists of two oppositely directed spikes in each branch of the dipole representing the rate of change of the current, Fig. 9. The electromagnetic fields set up in the radiation zone by the individual discontinuities in current largely cancel. A significant radiation field can be expected only in the short interval when the first spike has reached the antenna tip and is reflected

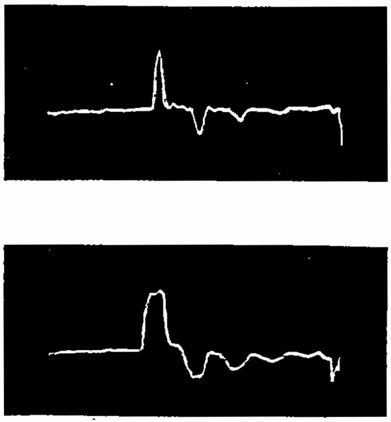
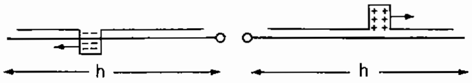
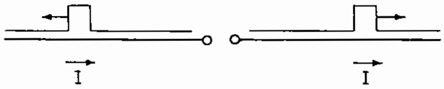


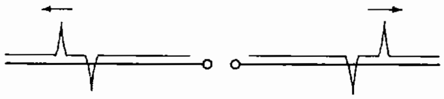
Fig. 8—Reflection of short pulses from circular loop antenna with half circumference $l = 9$ feet. Initial pulse 3 mμsec and 7 mμsec. Sweep rate 10 mμsec/cm.



(a) Charge distribution



(b) Current distribution



(c) Distribution of time derivative of current

Fig. 9—Radiation of short pulses from linear antenna. The arrows denote the direction of motion of the concentrations of charge.

in the opposite direction, while the second spike has not yet reached the tip. For short pulses, the radiation field should show a pattern of successive pulses, in alternating phase spaced at time intervals of $t_1 = 2h/c$. This is roughly in agreement with the results obtained theoretically and experimentally for an excitation by a step-function voltage, if the resultant transient field for a step function is differentiated in time to obtain the response to a delta-function input.[2]

CONCLUSION

It is shown that the initial transient response of linear antennas and loop antennas with regard to their circuit properties is approximately that of an infinitely long antenna near the highest frequency in the applied pulse. Successive reflections at the ends resemble those on open-ended or short-circuited sections of transmission line. The distortion of a short pulse traveling along the antenna is relatively small, even after a number of reflections from antenna tip and antenna base. If pulses of longer duration are transmitted, peak voltages in the transmission line can exceed the voltage of the applied pulse by a substantial amount. The response of a dipole to a step function has been analyzed by Wu.[7]

ACKNOWLEDGMENT

The authors are indebted to Prof. Tai Tsun Wu for illuminating discussions and valuable suggestions.

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[7] T. T. Wu, "Transient Response of a Dipole Antenna," Cruft Lab., Harvard University, Cambridge, Mass., Tech. Rept. No. 319, March, 1960; also in *J. Math. Phys.*, vol. 2, pp. 892–894; November-December, 1961.

A Method for Synthesis of Optimum Directional Patterns from Nonplanar Apertures*

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***Summary*—The scalar radiation problem is considered for sources lying on a surface of arbitrary shape. An optimum directional pattern is specified and the source distribution which gives rise to the optimum pattern is determined in terms of the Green's function for the surface. The optimum source distribution is found to be proportional to the complex conjugate of the Green's function evaluated at the beam pointing coordinates and to lie in a region enclosed by an equimagnitude curve of this function.**

Examples are included for sources on planes and spheres.

I. INTRODUCTION

THE STUDY OF radiation from surfaces of arbitrary shape represents an attempt to extend the elegance of synthesis techniques developed for linear and planar antennas to the general conformal case. This is not a simple task when the antenna shapes do not allow the approximate use of the Green's function for the infinite plane in the computation of the field produced by a known source. Indeed, many of the sophisticated linear and planar techniques are themselves limited to very special classes of radiators. The Dolph-Chebyshev source distribution, for example, which minimizes the beamwidth for a given sidelobe level, or vice versa, is applicable only to the case of discrete radiators which are spaced less than a wavelength apart.[1,2] Other optimum distributions suffer from similar restrictions.[3]

This report deals with the problem of synthesizing optimum directional patterns when the source exists on a radiating surface of arbitrary shape. The "optimum directional pattern" is defined here as that pattern having the greatest power density which can be produced in a specified direction by a source distribution limited in magnitude and extent in an integral sense.

In particular, the far-zone power density at a point in space is maximized by choosing the proper source distribution from among all those for which

$$\int \psi\psi^* dA = \text{constant},$$[4]

where ψ represents any source distribution and the integration is carried out over the entire surface which con-

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[†] Hughes Aircraft Company, Culver City, Calif.

[‡] American Systems, Inc., Hawthorne, Calif. Formerly with Hughes Aircraft Company.

[1] C. L. Dolph, "A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobe level," PROC. IRE., vol. 34, pp. 335–348; June, 1946.

[2] H. J. Riblet, Note on Dolph, *ibid.*, PROC. IRE, vol. 35, pp. 489–492; May, 1947.

[3] S. Silver, "Microwave Antenna Theory and Design," McGraw-Hill Book Co., Inc., New York, N. Y., p. 272; 1949.

[4] ψ^* denotes the complex conjugate of ψ.